Research Prospectus

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1 Introduction

My research falls within the area of *stable homotopy theory*, especially in the computational aspect. Stable homotopy theory is the subject that studies algebraic invariants of topological spaces that do not change under continuous deformation, which are also known as generalized (co)homology theories. All of the computations of these invariants can be reduced to the computation of stable homotopy groups.

1.1 Chromatic homotopy theory

Even for the sphere, the simplest CW complex, the computation of its stable homotopy groups is extremely difficult, and the result remains open. However, *chromatic homotopy theory* allows us to decompose this computation into the computation of the homotopy groups of the $K(n)$-local sphere $\pi_* L_{K(n)} S^0$, which are v_n-periodic, and therefore in principle computable in closed form.

Significant work has been done when n is 0 or 1: $L_{K(0)} S^0$ is the rationalization of the sphere, and $L_{K(1)} S^0$ is related to K-theory. For $n = 2$, the computation has been done by Shimomura-Yabe [SY95] at $p \geq 5$ and Shimomura-Wang [SW02] at $p = 3$, while the computation at the prime 2 is still open. I am currently working towards the computation at the prime 2 (c.f. §3.1). I am also investigating properties of Gross-Hopkins duality at $p \geq 5$ (c.f. §3.2).

1.2 Equivariant homotopy theory

Devinatz-Hopkins [DH04] realized $L_{K(n)} S^0$ as the homotopy fixed point spectrum $E_n^{h\mathbb{G}_n}$, where E_n is a spectrum admitting a $\mathbb{G}_n$-action. *Equivariant homotopy theory* is the homotopy theory studying topological spaces with continuous group actions. It has a wide range of applications, the most representative one among which is Hill-Hopkins-Ravenel's solution [HHR16] to the Kervaire invariant one problem, which involved studying the C_8-equivariant spectrum $\Omega_{\mathbb{O}}$.

My work [Mab] focuses on C_2-equivariant homotopy theory (c.f. §2.1). In [Mab], I compared two methods of computing C_2-equivariant stable homotopy groups, and related them to the computation of classical stable homotopy groups. I am seeking a generalization to the C_p-equivariant case, where p is an odd prime (c.f. §3.3).

1.3 Motivic homotopy theory

S-motivic homotopy theory is the homotopy theory of smooth schemes of finite type over S. Gheorghe-Wang-Xu [GWX21] revealed an astonishing relation between $\mathbb{C}$-motivic stable homo-

topy theory and classical stable homotopy theory, which Isaksen-Wang-Xu [IWX23] was able to use to extend the range of the classical computation by 50%. By considering the action of the Galois group, Behrens-Shah [BS20] related $\mathbb{R}$-motivic homotopy theory with C_2-equivariant homotopy theory, which is the main technique in dealing with the C_2-equivariant computation in [Mab].

1.4 Synthetic spectra

One year after [GWX21], Gheorghe-Isaksen-Krause-Ricka [GIKR22] used filtered spectra to give an alternate description of 2-completed cellular $\mathbb{C}$-motivic spectra in an algebraic topological way. This was further generalized to synthetic spectra by Pstrągowski in [Pst23].

Filtered spectra and synthetic spectra are powerful tools when we need to compare differentials in different spectral sequences. In my work [Ma24b], I used them to prove a generalized geometric boundary theorem, which transports information about Adams spectral sequences for different spectra along the maps in a cofiber sequence (c.f. §2.2). Synthetic spectra are also used in my current work on chromatic computations (c.f. §3.1).

1.5 Summary

I have broad research interests in all these aspects of stable homotopy theory, which interact with each other in a beautiful way.

2 Previous projects

2.1 C_2-equivariant Adams spectral sequences [Mab]

In this subsection, we will focus at the prime 2, and every object is assumed to be 2-completed.

One of the most efficient ways to compute the classical stable homotopy homotopy groups of spheres is the Adams spectral sequence:

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*\mathbb{S}.$$

In the C_2-equivariant context, two different Adams spectral sequences have been constructed. Greenlees [Gre85, Gre88, Gre90] constructed the Borel C_2-equivariant Adams spectral sequence:

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^h_{*,*}}\left(\left(H\underline{\mathbb{F}_2}\right)^h_{*,*}, \left(H\underline{\mathbb{F}_2}\right)^h_{*,*}\right) \Rightarrow \pi^{C_2}_{*,*}\left(\mathbb{S}_{\mathrm{C}_2}\right)^h.$$

Hu-Kriz [HK01] constructed the genuine C_2-equivariant Adams spectral sequence:

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{C_2}_{*,*}}\left(\left(H\underline{\mathbb{F}_2}\right)_{*,*}, \left(H\underline{\mathbb{F}_2}\right)_{*,*}\right) \Rightarrow \pi^{C_2}_{*,*}\mathbb{S}_{\mathrm{C}_2}.$$

By the Segal conjecture for C_2 [Lin80], the map

$$\mathbb{S}_{\mathrm{C}_2} \to \left(\mathbb{S}_{\mathrm{C}_2}\right)^h$$

is a (2-adic) equivalence. Furthermore, there is a canonical map from the genuine equivariant Adams spectral sequence to the Borel Adams spectral sequences. Therefore, it is natural to ask:

Question 1. *Are these two spectral sequences isomorphic?*

By the change-of-rings isomorphism, the map on the E_2-terms is the map

$$\mathrm{Ext}^{*,*,*}_{\mathcal{A}^{C_2}_{*,*}}\left(\left(H\underline{\mathbb{F}_2}\right)_{*,*}, \left(H\underline{\mathbb{F}_2}\right)_{*,*}\right) \to \mathrm{Ext}^{*,*,*}_{\mathcal{A}^{C_2}_{*,*}}\left(\left(H\underline{\mathbb{F}_2}\right)_{*,*}, \left(H\underline{\mathbb{F}_2}\right)^h_{*,*}\right)$$

obtained by applying the Ext functor to the map

$$\left(H\underline{\mathbb{F}_2}\right)_{*,*} \to \left(H\underline{\mathbb{F}_2}\right)^h_{*,*}. \tag{i}$$

Since map (i) is neither injective nor surjective, we cannot get a long exact sequence directly. An idea is to factor this map through its image, but the problem is that the factorization does not lift to a factorization of spectral sequences in the C_2-equivariant context.

To solve this problem, I used $\mathbb{R}$-motivic homotopy theory, since the homotopy group of the $\mathbb{R}$-motivic Eilenberg-MacLane spectrum $\mathbb{M}_2$ is isomorphic to the image of (i). Using homological algebra, I was finally able to give a negative answer to Question 1 by showing

Theorem 2 ([Mab, Thm. 6.2])**.** *The E_2-term of the Borel Adams spectral sequence can be obtained from the genuine one by shifting the filtration degree of the elements in the negative cone up by* 1 *and cancelling the "shortened" genuine d_2 differentials.*

By more detailed analysis, I was able to compare the differentials in these two spectral sequences:

Theorem 3 ([Mab, Thm. 6.3, 6.4])**.** *The differentials in the Borel and genuine Adams spectral sequences can be deduced from each other up to indeterminacy.*

On the other hand, I also studied the computation of the Borel Adams spectral sequence. Using the algebraic Lin's theorem [LDMA80], I was able to show

Theorem 4 ([Mab, Thm. 4.1, 4.2])**.** *The Borel Adams spectral sequence is isomorphic to the direct sum of at most two classical Adams spectral sequences. The isomorphism of the E_2-terms is given by*

$$\mathrm{Ext}^{s,t,w}_{\mathcal{A}^h_{*,*}}\left(\left(H\underline{\mathbb{F}_2}\right)^h_{*,*}, \left(H\underline{\mathbb{F}_2}\right)^h_{*,*}\right) \cong \begin{cases} \mathrm{Ext}^{s-1,t-w-1}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-w-1}_{-\infty}\right), & w > 0; \\ \mathrm{Ext}^{s-1,t-w-1}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_{-w}\right) \oplus \mathrm{Ext}^{s,t-w-1}_{\mathcal{A}_*}\left(H_*S^{-1}\right), & w \leq 0. \end{cases}$$

Since the $\mathcal{A}_*$-comodules appearing in the right-hand side are bounded below, the Ext groups can be computed using the Curtis algorithm. I wrote a program [Ma24a] to process the computation, through which I was able to produce charts of the E_2-term of the Borel Adams spectral sequence as shown in [Mab, Appx. A].

This computation should be compared with Guillou-Isaksen's computation [GI] of the genuine C_2-equivariant Adams spectral sequence. Furthermore, a partial comparison between the naming systems of [GI] and [Mab] was given in [Mab, Sec. 7].

2.2 Generalized geometric boundary theorem [Ma24b]

When analysing the spectral sequences in [Mab], I need to compare the Adams spectral sequences of spectra in a cofiber sequence. This problem is known as the generalized geometric boundary theorem, and was proved in [Beh12, Lem. A.4.1]. However, I noticed that its assumption is not sufficient to support the result. This motivated me to write [Ma24b] to fix the mistake and make the proof more readable.

When comparing the differentials that occur in different pages of spectral sequences, I made use of the power of filtered spectra. Filtered spectra translate all spectral sequences into τ-Bockstein spectral sequences, and the structure of the homotopy groups as $\mathbb{Z}[\tau]$-modules contains the information of differentials in different pages simultaneously. Using filtered spectra, I was able to formulate the correct assumptions for the generalized geomeric boundary theorem and write a cleaner proof (c.f. [Ma24b, Thm. 1, 2, 3, 4, 5]).

3 Ongoing and future projects

In this section, I will discuss the subject of my thesis [Mac], which invloves a computation of the $K(2)$-local homotopy of a finite complex $Y_{\mathbb{H}}$ at the prime 2. I will also discuss the next steps in this project, as well as other projects I plan to pursue.

3.1 The $K(2)$-local homotopy groups of $Y_{\mathbb{H}}$ at $p = 2$

In [BBB$^+$21], the $K(2)$-local homotopy groups of Z at the prime 2 have been computed up to a possible extension, where Z is a 2-local finite spectrum of type 2 with 32 cells. On the other hand, the $K(2)$-local homotopy groups of the sphere at the prime 2 are still too complicated to compute. It is a natural idea to study an intermediate object to connect the computation of S^0 and Z.

The object I am considering is $Y_{\mathbb{H}} \simeq \Sigma^{-7}\mathbb{RP}^2 \wedge \mathbb{CP}^2 \wedge \mathbb{HP}^2$, an 8-cell spectrum of type 1. It has two advantages over Pham's choice A_1 in his thesis [Pha19]: $Y_{\mathbb{H}}$ has no ambiguity, while there are four versions of A_1; the computation of $Y_{\mathbb{H}}$ includes v_1-local information as it is of type 1, while A_1 does not as it is of type 2.

Note that we have $\mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2$ and $\mathbb{G}_2^1 \cong \mathbb{S}_2^1 \rtimes C_2$. To compute the homotopy groups of $L_{K(2)}Y_{\mathbb{H}} \simeq E_2^{h\mathbb{G}_2} \wedge Y_{\mathbb{H}}$, one first needs to compute the homotopy groups of $E_2^{h\mathbb{S}_2^1} \wedge Y_{\mathbb{H}}$. The tool I use is the topological duality spectral sequence (TDSS) induced by the finite resolution

$$E_2^{h\mathbb{S}_2^1} \wedge Y_{\mathbb{H}} \to E_2^{hG_{24}} \wedge Y_{\mathbb{H}} \to E_2^{hC_6} \wedge Y_{\mathbb{H}} \to \Sigma^{48}E_2^{hC_6} \wedge Y_{\mathbb{H}} \to \Sigma^{48}E_2^{hG_{24}} \wedge Y_{\mathbb{H}}$$

constructed by Bobkova-Goerss [BG18]. Converging to the homotopy groups of each term, there are the Adams-Novikov spectral sequences (ANSS) whose E_2-terms form the algebraic duality spectral sequence (ADSS) converging to $H^*(\mathbb{S}_2^1; (E_2)_*Y_{\mathbb{H}})$, which is constructed by Beaudry [Bea16].

I have completely computed the ADSS for $Y_{\mathbb{H}}$.

Theorem 5 ([Mac])**.** *The differentials in E_1-page of the ADSS for $Y_{\mathbb{H}}$ are completely computed. The ADSS for $Y_{\mathbb{H}}$ collapses at the E_2-page.*

Furthermore, using the MU-synthetic method, I am able to combine the ADSS and the ANSS together, and get the following result on the TDSS for $Y_{\mathbb{H}}$.

Theorem 6 ([Mac])**.** *The differentials in E_1-page of the TDSS for $Y_{\mathbb{H}}$ are completely computed. There are no nontrivial differentials in the E_2-page.*

I am optimistic of the following conjecture.

Conjecture 7. *There are no nontrivial differentials in the E_3-page of the TDSS for $Y_{\mathbb{H}}$, and hence the TDSS collapses at the E_2-page.*

To prove this conjecture, I plan to compute the tmf-Hurewicz image of $Y_{\mathbb{H}}$ by the tmf-resolution. If the conjecture holds, the TDSS for $Y_{\mathbb{H}}$ would be completely computed, which gives us the homotopy groups of $E_2^{h\mathbb{S}_2^1} \wedge Y_{\mathbb{H}}$. After that, I will endeavor to compute the homotopy groups of $L_{K(2)}Y_{\mathbb{H}}$, and will apply this computation to study the computation of the $K(2)$-local homotopy groups of spheres at $p = 2$.

3.2 Homotopy groups of $E_2^{h\mathbb{G}_2^1}$ at $p \geq 5$

Although we have known the homotopy groups of $L_{K(2)}S^0$ as abelian groups at $p \geq 5$, we do not understand the ring structure very well. In particular, there is an element $\zeta \in \pi_{-1}L_{K(2)}S^0$ with $\zeta^2 = 0$. We do not know the $\Lambda_{\mathbb{F}_p}[\zeta]$-module structure on $\pi_*L_{K(2)}S^0$. Note that the semidirect product decomposition $\mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_p$ induces the cofiber sequence

$$L_{K(2)}S^0 \to E_2^{h\mathbb{G}_2^1} \xrightarrow{\psi-1} E_2^{h\mathbb{G}_2^1},$$

and ζ is the composite $S^{-1} \to \Sigma^{-1}E_2^{h\mathbb{G}_2^1} \to L_{K(2)}S^0$. Therefore, it is important to understand the homotopy groups of $E_2^{h\mathbb{G}_2^1}$. Such results might also be important for the computation at $p = 2$.

While the computation is tough, we can reduce the amount of computation by considering the Gross-Hopkins dual $I_2X \simeq D_2X \wedge \Sigma^2 S[det]$ ([HG94]), where $X \in Sp_{K(2)}$ and $S[det]$ is an element in $\mathrm{Pic}(Sp_{K(2)})$. It is hard to apply directly though, since we do not know the homotopy groups of $S[det]$. However, I have proved that $E_2^{h\mathbb{G}_2^1}$ is Gross-Hopkins self dual up to a p-adic suspension:

Theorem 8 ([Maa, Cor. 4])**.** *There is an equivalence*

$$I_2E_2^{h\mathbb{G}_2^1} \simeq \Sigma^{(1+p+p^2+\cdots)|v_2|+2p+3}E_2^{h\mathbb{G}_2^1}.$$

I am seeking the application of this self-duality result to the computations.

3.3 C_p-equivariant Adams spectral sequences for odd primes

There is a fair amount of computations of the C_2-equivariant homotopy groups of spheres using the genuine equivariant Adams spectral sequence (c.f. [GI]). On the other hand, the C_p-equivariant dual Steenrod algebra for odd primes was computed by Hu-Kriz-Somberg-Zou [HKSZ]. However, the result is complicated and its cohomology, which is the E_2-term of the genuine equivariant Adams spectral sequence, is hard to compute.

In [Mab], I have shown that the genuine C_2-equivariant Adams spectral sequence is related to the Borel Adams spectral sequence, which, by Theorem 4, is isomorphic to the classical Adams spectral sequences of the stunted real projective spectra (c.f. §2.1). I am interested in its generalization to the C_p-equivariant case for odd primes, so that it would be possible to understand the genuine C_p-equivariant Adams spectral sequence through the classical Adams spectral sequences of the stunted lens spaces.

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