SELF-DUALITY OF $E_2^{h\mathbb{G}_2^1}$ AT $p \geq 5$

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Abstract. In this note, we will show that the Gross-Hopkins dual of $E_2^{h\mathbb{G}_2^1}$ is a p-adic suspension of itself at $p \geq 5$.

Recall that we have the determinant map
$$\det : \mathbb{G}_2 \to \mathbb{Z}_p^\times.$$
Composing it with the quotient map $\mathbb{Z}_p^\times/\mathbb{F}_p^\times \cong \mathbb{Z}_p$ gives a homomorphism
$$\zeta_2 : \mathbb{G}_2 \to \mathbb{Z}_p.$$
Denote by $\mathbb{G}_2^1$ the kernel of ζ_2 and let $\overline{S} = E_2^{h\mathbb{G}_2^1}$.

For a Morava module $M \in \mathrm{Mod}_{(E_2)_*}^{\mathbb{G}_2}$, let $M[\det^k] \in \mathrm{Mod}_{(E_2)_*}^{\mathbb{G}_2}$ be the Morava module twisted by $\det^k : \mathbb{G}_2 \to \mathbb{Z}_p^\times \subset (E_2)_0^\times$. To be explicit, the twisted $\mathbb{G}_2$ action on $M[\det^k]$ is given by
$$g_{\det^k} m = \det(g)^k gm.$$
Recall from the unpublished result of Hopkins that
$$\mathrm{Pic}(Sp_{K(2)}) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}/2(p^2-1)$$
is topologically generated by $L_{K(2)}S^1$ and $S[\det]$. The isomorphism can be chosen such that $L_{K(2)}S^1$ and $S[\det]$ correspond to $(1,0,1)$ and $(0,1,2(p+1))$ respectively. The determinant sphere $S[\det]$ satisfies that $(E_2)_*^\vee S[\det] \cong (E_2)_*[\det]$ as Morava modules.

Lemma 1. *There is an isomorphism of Morava modules*
$$(E_2)_*^\vee \overline{S} \cong (E_2)_*^\vee \overline{S}\left[\det^{p-1}\right].$$

Proof. By [DH04, Thm. 2],
$$(E_2)_*^\vee \overline{S} \cong \mathrm{Map}^c\left(\mathbb{G}_2/\mathbb{G}_2^1, (E_2)_*\right) \cong \mathrm{Map}^c\left(\mathbb{Z}_p, (E_2)_*\right),$$
where the $\mathbb{G}_2$-action is given by
$$(g\phi)(x) = g\phi(g^{-1}x) = g\phi(x - \zeta_2(g)).$$
Note that there is a split short exact sequence
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{F}_p^\times \arrow[r, shift left] & \mathbb{Z}_p^\times \arrow[l, shift left, "\bmod p"] \arrow[r, shift left] & \mathbb{Z}_p \arrow[l, shift left, "e^{px}"] \arrow[r] & 0 \end{tikzcd}.$$
We then claim that for any $g \in \mathbb{G}_2$,
$$\det(g)^{p-1} = \left(e^{p\zeta_2(g)}\right)^{p-1} \in \mathbb{Z}_p^\times.$$
This is because $\det(g)$ and $e^{p\zeta_2(g)}$ have the same image $\zeta_2(g)$ in $\mathbb{Z}_p$. After taking $(p-1)^{st}$ power, both of them are congruent to 1 mod p.

Now we construct the following map

$$F : \mathrm{Map}^c(\mathbb{Z}_p, (E_2)_*) \to \mathrm{Map}^c(\mathbb{Z}_p, (E_2)_*)[\det^{p-1}];$$
$$\phi \mapsto \left(F(\phi) : x \mapsto e^{(p^2-p)x}\phi(x)\right).$$

Note that

$$F(g\phi)(x) = e^{(p^2-p)x}(g\phi)(x) = e^{(p^2-p)x} g\phi(x - \zeta_2(g)),$$

and that

$$\begin{aligned}
(g_{\det^{p-1}}F)(\phi)(x) &= \det(g)^{p-1} gF(\phi)(x-\zeta_2(g)) \\
&= e^{(p^2-p)\zeta_2(g)} g\left(e^{(p^2-p)(x-\zeta_2(g))}\phi(x-\zeta_2(g))\right) \\
&= e^{(p^2-p)\zeta_2(g)} e^{(p^2-p)(x-\zeta_2(g))} g\phi(x-\zeta_2(g)) \\
&= e^{(p^2-p)x} g\phi(x-\zeta_2(g)),
\end{aligned}$$

we can see that F is a map of Morava modules. Similarly, we construct the map

$$G : \mathrm{Map}^c(\mathbb{Z}_p, (E_2)_*)[\det^{p-1}] \to \mathrm{Map}^c(\mathbb{Z}_p, (E_2)_*);$$
$$\psi \mapsto \left(G(\psi) : x \mapsto e^{(p-p^2)x}\psi(x)\right).$$

Note that

$$\begin{aligned}
G(g_{\det^{p-1}}\psi)(x) &= e^{(p-p^2)x}\det(g)^{p-1} g\psi(x-\zeta_2(g)) \\
&= e^{(p-p^2)(x-\zeta_2(g))} g\psi(x-\zeta_2(g)) \\
&= g\left(e^{(p-p^2)(x-\zeta_2(g))}\psi(x-\zeta_2(g))\right) \\
&= (gG)(\psi)(x),
\end{aligned}$$

we can see that G is also a map of Morava modules. It is easy to see that F and G are inverses of each other, and the result follows. □

Lemma 2. *The functor $(E_2)^\vee_*(-)$ induces an isomorphism*

$$\pi_0 \mathrm{Map}\left(\overline{S}, \overline{S}[\det^{p-1}]\right) \to \mathrm{Hom}_{\mathrm{Mod}^{\mathbb{G}_2}_{(E_2)_*}}\left((E_2)^\vee_*\overline{S}, (E_2)^\vee_*\overline{S}\left[\det^{p-1}\right]\right).$$

Proof. By [Hov04, Thm. 2.6], $(E_2)^\vee_*\overline{S} \cong \mathrm{Map}^c(\mathbb{Z}_p, (E_2)_*)$ is pro-free. By [BH16, Thm. 3.1], there is a $K(2)$-local E_2-Adams spectral sequence

$$E_2^{s,t} = \widehat{\mathrm{Ext}}^{s,t}_{(E_2)^\vee_* E_2}\left((E_2)^\vee_* E_2^{h\mathbb{G}_2^1}, (E_2)^\vee_* E_2^{h\mathbb{G}_2^1}\right) \Rightarrow \pi_{t-s}\mathrm{Map}\left(\overline{S}, \overline{S}[\det^{p-1}]\right). \qquad (1)$$

By [BH16, Cor. 3.2], there is an isomorphism

$$\widehat{\mathrm{Ext}}^{s,t}_{(E_2)^\vee_* E_2}\left((E_2)^\vee_* E_2^{h\mathbb{G}_2^1}, (E_2)^\vee_* E_2^{h\mathbb{G}_2^1}\right) \cong H^s_c\left(\mathbb{G}_2^1, \pi_t \mathrm{Map}(\overline{S}, E_2)\right).$$

By [GHMR05, Prop. 2.5], there is an isomorphism

$$\pi_* \mathrm{Map}(\overline{S}, E_2) = (E_2)_*[[\mathbb{Z}_p]].$$

Then the E_2-term of the spectral sequence (1) becomes

$$E_2^{s,t} = H^s_c\left(\mathbb{G}_2^1; (E_2)_t[[\mathbb{Z}_p]]\right).$$

By [Hen07, Thm. 6], $\mathrm{cd}_p(\mathbb{G}_2^1) \leq 3$, and thus, $E_2^{s,t} = 0$ for $s > 3$. On the other hand, for any $a \in \mathbb{F}_p^\times \subset \mathbb{Z}_p^\times \subset \mathbb{G}_2$, we have $\det(a) = a^2$, and hence $\zeta_2(a) = 0$. Therefore, $\mathbb{F}_p^\times$ is a subgroup of $\mathbb{G}_2^1$. Furthermore, it is the torsion subgroup of $\mathbb{Z}_p^\times$, the center of $\mathbb{G}_2$. This implies that $\mathbb{F}_p^\times$ is a normal subgroup of $\mathbb{G}_2$, so it is also a normal subgroup of $\mathbb{G}_2^1$. Then we have the Hochschild-Serre spectral sequence

$$H_c^*\left(\mathbb{G}_2^1/\mathbb{F}_p^\times; H^*(\mathbb{F}_p^\times; (E_2)_*[[\mathbb{Z}_p]])\right) \Rightarrow H_c^*\left(\mathbb{G}_2^1; (E_2)_*[[\mathbb{Z}_p]]\right).$$

Note that $p-1$, the order of $\mathbb{F}_p^\times$, is invertible in $(E_2)_*[[\mathbb{Z}_p]]$, the cohomology $H^*(\mathbb{F}_p^\times; (E_2)_*[[\mathbb{Z}_p]]) \cong (E_2)_*[[\mathbb{Z}_p]]^{\mathbb{F}_p^\times}$ is concentrated at cohomological degree 0. Thus, the Hochschild-Serre spectral sequence collapses, and we have

$$H_c^s\left(\mathbb{G}_2^1; (E_2)_t[[\mathbb{Z}_p]]\right) \cong H_c^s\left(\mathbb{G}_2^1/\mathbb{F}_p^\times; (E_2)_t[[\mathbb{Z}_p]]^{\mathbb{F}_p^\times}\right).$$

Here $\mathbb{F}_p^\times$ acts on $\mathbb{Z}_p \cong \mathbb{G}_2/\mathbb{G}_2^1$ trivially. For $a \in \mathbb{F}_p^\times$, its action on $(E_2)_{2t}$ is the multiplication by a^t. Therefore, $H_c^s\left(\mathbb{G}_2^1; (E_2)_t[[\mathbb{Z}_p]]\right)$ is only possibly nontrivial if t is a multiple of $2(p-1)$. The sparseness then implies that

$$\begin{aligned}
\pi_0 \operatorname{Map}\left(\overline{S}, \overline{S}[\det^{p-1}]\right) &\cong \widehat{\operatorname{Ext}}^{0,0}_{(E_2)_*^\vee E_2}\left((E_2)_*^\vee E_2^{h\mathbb{G}_2^1}, (E_2)_*^\vee E_2^{h\mathbb{G}_2^1}\right) \\
&\cong \operatorname{Hom}_{\widehat{\operatorname{Comod}}_{(E_2)_*^\vee E_2}}\left((E_2)_*^\vee E_2^{h\mathbb{G}_2^1}, (E_2)_*^\vee E_2^{h\mathbb{G}_2^1}\right) \\
&\cong \operatorname{Hom}_{\operatorname{Mod}^{\mathbb{G}_2}_{(E_2)_*}}\left((E_2)_*^\vee E_2^{h\mathbb{G}_2^1}, (E_2)_*^\vee E_2^{h\mathbb{G}_2^1}\right),
\end{aligned}$$

where the last isomorphism is due to [BH16, Cor. 5.5]. Then the result follows. ☐

Theorem 3. *There is an equivalence*

$$\overline{S}\left[\det^{p-1}\right] \simeq \overline{S}.$$

Proof. By Lemma 2, there is a map $\overline{S} \to \overline{S}[\det^{p-1}]$ realizing the isomorphism in Lemma 1. Then it induces an equivalence

$$L_{K(2)}\left(E_2 \wedge \overline{S}\right) \xrightarrow{\simeq} L_{K(2)}\left(E_2 \wedge \overline{S}[\det^{p-1}]\right).$$

By [HS99, Thm. 8.9], this map itself is an equivalence. ☐

Corollary 4. *There is an equivalence*

$$I_2\overline{S} \simeq \Sigma^{(1+p+p^2+\cdots)|v_2|+2p+3}\overline{S}.$$

Proof.

$$\begin{aligned}
I_2\overline{S} &\simeq D_2\overline{S} \wedge I_2 \\
&\simeq \Sigma^{-1}\overline{S} \wedge I_2 \\
&\simeq \Sigma^{-1}\overline{S} \wedge S^2[\det] \\
&\simeq \Sigma^{(1+p+p^2+\cdots)|v_2|+2p+3}\overline{S}.
\end{aligned}$$

☐

References

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