This is a program to compute the $\mathbb{F}_2[\rho]$-module structure on the $E_2$-term of the Borel $C_2$-equivariant Adams spectral sequence for the $C_2$-equivariant sphere:

$$E_2^{s,t,w} = \mathrm{Ext}^{s,t,w}_{\mathcal{A}^h_{*,*}}\left(\left(H_{C_2}\right)^h_{*,*}\right) \cong \begin{cases} \mathrm{Ext}^{s-1,t-w-1}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-w-1}_{-\infty}\right), & w>0; \\ \mathrm{Ext}^{s-1,t-w-1}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_{-w}\right) \oplus \mathrm{Ext}^{s,t-w-1}_{\mathcal{A}_*}\left(H_*S^{-1}\right), & w \leq 0. \end{cases}$$

The file `AAHSS_without_-1.exe` uses the Curtis algorithm to compute the Lambda complex of $\Sigma\mathbb{P}^{-w-1}_{-\infty}$ for $w>0$, where

$$H_*\Sigma\mathbb{P}^{-w-1}_{-\infty} = \begin{cases} \mathbb{F}_2, & * \geq -w \text{ and } * \neq -1, \\ 0, & \text{otherwise.} \end{cases}$$

After inputting a positive integer $w$ as the weight, and a positive integer $t$ as the total degree, you will get an output of all permanent cycles and differentials between elements from different cells in the Lambda complex, ordered by the homological degree $s$.

For example, when you input $w=6$ and $t=12$, you will get the output, a part of which is listed as follows:

```
homological␣degree␣s=3␣(topological␣degree␣9,␣coweight␣3):
3␣0␣0␣<--␣4␣0
1␣1␣1
-3␣3␣3␣<--␣1␣3
-5␣5␣3␣<--␣-3␣7

homological␣degree␣s=4␣(topological␣degree␣8,␣coweight␣2):
-5␣5␣1␣1␣<--␣-4␣6␣1
-6␣2␣3␣3␣<--␣0␣2␣1
```

The line

```
1␣1␣1
```

implies that there is an element

$$h_1^2[1] \in \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-7}_{-\infty}\right),$$

which maps to the nontrivial element

$$h_1^2[1] \in \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-2}_{-\infty}\right).$$

The line

```
-6␣2␣3␣3␣<--␣0␣2␣1
```

implies that there is an element

$$h_0h_2[0] \in \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-6}_{-\infty}\right),$$

which maps to the nontrivial element

$$h_0h_2[0] \in \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-2}_{-\infty}\right),$$

but cannot be lifted to $\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-7}_{-\infty}\right)$.
The line

```
-5␣5␣1␣1␣<--␣-4␣6␣1
```

implies that there is an element

$$h_0h_3[-4] \in \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-5}_{-\infty}\right),$$

which maps to 0 in $\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-4}_{-\infty}\right)$, and cannot be lifted to $\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-6}_{-\infty}\right)$.

The file `AAHSS_with_-1.exe` computes the Lambda complex of $\mathbb{P}^{\infty}_{-w}$.

For example, when you input $w = 6$ and $t = 12$, you will get the output, a part of which is listed as follows:

```
homological␣degree␣s=3␣(topological␣degree␣9,␣coweight␣3):
3␣0␣0␣<--␣4␣0
-3␣3␣3␣<--␣1␣3
-5␣5␣3␣<--␣-3␣7

homological␣degree␣s=4␣(topological␣degree␣8,␣coweight␣2):
-1␣1␣1␣1␣<--␣0␣2␣1
-5␣5␣1␣1␣<--␣-4␣6␣1
-6␣2␣3␣3␣<--␣1␣1␣1
```

The line

```
-1␣1␣1␣1␣<--␣0␣2␣1
```

implies that there is an element

$$h_0h_2[0] \in \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_{0}\right),$$

which maps to 0 in $\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_{1}\right)$.

The line

```
-6␣2␣3␣3␣<--␣1␣1␣1
```

implies that there is an element

$$h_1^2[1] \in \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_{1}\right),$$

which can be lifted to

$$h_1^2[1] \in \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_{0}\right),$$

and maps to 0 in $\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_{2}\right)$.

To understand the $\rho$-multiplication map

$$\mathrm{Ext}^{3,7,1}_{\mathcal{A}^h_{*,*}}\left(\left(H_{C_2}\right)^h_{*,*}\right) \to \mathrm{Ext}^{3,6,0}_{\mathcal{A}^h_{*,*}}\left(\left(H_{C_2}\right)^h_{*,*}\right),$$

we need to extend the range by inputting $w = 7$ and $t = 13$ to get the partial output from `AAHSS_without_-1.exe`:

```
homological␣degree␣s=3␣(topological␣degree␣10,␣coweight␣3):
3␣0␣0␣<--␣4␣0
-3␣3␣3␣<--␣1␣3
-5␣5␣3␣<--␣-3␣7

homological␣degree␣s=4␣(topological␣degree␣9,␣coweight␣2):
-5␣5␣1␣1␣<--␣-4␣6␣1
-6␣2␣3␣3␣<--␣0␣2␣1
-7␣3␣3␣3␣<--␣1␣1␣1
```

and the partial output from `AAHSS_with_-1.exe`:

```
homological␣degree␣s=3␣(topological␣degree␣10,␣coweight␣3):
3␣0␣0␣<--␣4␣0
-3␣3␣3␣<--␣1␣3
-5␣5␣3␣<--␣-3␣7

homological␣degree␣s=4␣(topological␣degree␣9,␣coweight␣2):
-1␣1␣1␣1␣<--␣0␣2␣1
-5␣5␣1␣1␣<--␣-4␣6␣1
-6␣2␣3␣3␣<--␣1␣1␣1
-7␣3␣3␣3
```

Then we can see that $h_0h_2[0]$ maps to $c_0[-6]$ under the map

$$\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-2}_{-\infty}\right) \to \varprojlim_k \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{-2}_{-k}\right) \to \varprojlim_k \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{-1}_{-k}\right).$$

On the other hand, $c_0[-6]$ is the image of $h_1^2[1]$ under the map

$$\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_0\right) \to \varprojlim_k \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{-1}_{-k}\right).$$

Therefore, $h_0h_2[0]$ maps to $h_1^2[1]$ under the upper map in the square

$$\begin{tikzcd}
\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-2}_{-\infty}\right) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_0\right) \oplus \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*S^{-1}\right) \arrow[d, "\cong"] \\
\mathrm{Ext}^{3,7,1}_{\mathcal{A}^h_{*,*}}\left((H_{C_2})^h_{*,*}\right) \arrow[r, "\cdot\rho"] & \mathrm{Ext}^{3,6,0}_{\mathcal{A}^h_{*,*}}\left((H_{C_2})^h_{*,*}\right).
\end{tikzcd}$$

We can also see that $h_1^2[1]$ maps to $h_2^3[-7]$ under the map

$$\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-2}_{-\infty}\right) \to \varprojlim_k \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{-2}_{-k}\right) \to \varprojlim_k \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{-1}_{-k}\right).$$

On the other hand, $h_2^3[-7]$ is not in the image of the map

$$\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_0\right) \to \varprojlim_k \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{-1}_{-k}\right).$$

Therefore, it maps to a nontrivial element under the map

$$\varprojlim_k \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{-1}_{-k}\right) \to \varprojlim_k \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_{-k}\right) \cong \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*S^{-1}\right),$$

which has to be $h_1^3[-1]$. Therefore, $h_1^2[1]$ maps to $h_1^3[-1]$ under the upper map in the square

$$\begin{tikzcd}
\mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\Sigma\mathbb{P}^{-2}_{-\infty}\right) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^{2,5}_{\mathcal{A}_*}\left(H_*\mathbb{P}^{\infty}_0\right) \oplus \mathrm{Ext}^{3,5}_{\mathcal{A}_*}\left(H_*S^{-1}\right) \arrow[d, "\cong"] \\
\mathrm{Ext}^{3,7,1}_{\mathcal{A}^h_{*,*}}\left((H_{C_2})^h_{*,*}\right) \arrow[r, "\cdot\rho"] & \mathrm{Ext}^{3,6,0}_{\mathcal{A}^h_{*,*}}\left((H_{C_2})^h_{*,*}\right).
\end{tikzcd}$$